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# Some geometric properties of hypersurfaces with constant $r$ -mean curvature in Euclidean space

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## Abstract

Let  $f : M \rightarrow \mathbb{R}^{m+1}$  be an isometrically immersed hypersurface. In this paper, we exploit recent results due to the authors in [4] to analyze the stability of the differential operator  $L_r$  associated with the  $r$ -th Newton tensor of  $f$ . This appears in the Jacobi operator for the variational problem of minimizing the  $r$ -mean curvature  $H_r$ . Two natural applications are found. The first one ensures that, under a mild condition on the integral of  $H_r$  over geodesic spheres, the Gauss map meets each equator of  $\mathbb{S}^m$  infinitely many times. The second one deals with hypersurfaces with zero  $(r + 1)$ -mean curvature. Under similar growth assumptions, we prove that the affine tangent spaces  $f_*T_pM$ ,  $p \in M$ , fill the whole  $\mathbb{R}^{m+1}$ .

## 1 Introduction

In what follows  $f : M^m \rightarrow \mathbb{R}^{m+1}$  will always denote a connected, orientable, complete, non compact hypersurface of Euclidean space. We fix an origin  $o \in M$  and let  $r(x) = \text{dist}(x, o)$ ,  $x \in M$ . We set  $B_r$  and  $\partial B_r$  for, respectively, the geodesic ball and the geodesic sphere centered at  $o$  with radius  $r$ . Moreover, let  $\nu$  be the spherical Gauss map and denote with  $A$  both the second fundamental form and the shape operator in the orientation of  $\nu$ . Associated with  $A$  we have the principal curvatures  $k_1, \dots, k_m$  and the set of symmetric functions  $S_j$ :

$$S_j = \sum_{i_1 < i_2 < \dots < i_j} k_{i_1} \cdot k_{i_2} \cdot \dots \cdot k_{i_j}, \quad j \in \{1, \dots, m\}, \quad S_0 = 1.$$

The  $j$ -mean curvature of  $f$  is defined

$$H_0 = 1, \quad \binom{n}{j} H_j = S_j,$$

so that, for instance,  $H_1$  is the mean curvature and  $H_m$  is the Gauss-Kronecker curvature of the hypersurface. Note that, when changing the orientation  $\nu$ , the odd curvatures change sign, while the sign of the even curvatures is an invariant of the immersion. By Gauss equations and flatness of  $\mathbb{R}^{m+1}$  it is easy to see that

$$H_2 = \binom{m}{2}^{-1} S_2 = \frac{1}{2} \binom{m}{2}^{-1} \text{scal},$$

where  $\text{scal}$  is the scalar curvature of  $M$ . The  $j$ -mean curvatures satisfy the so-called Newton inequalities

$$H_j^2 \geq H_{j-1} H_{j+1},$$

equality holding if and only if  $p$  is an umbilical point (see [9]). We stress that no restriction is made on the sign of the  $H_i$ 's.

**Theorem 1.1.** *Let  $f : M \rightarrow \mathbb{R}^{m+1}$  be a hypersurface such that, for some  $j \in \{0, m-2\}$ ,  $H_{j+1}$  is a non-zero constant. If  $j \geq 1$ , assume that there exists a point  $p \in M$  at which the second fundamental form is definite. Set*

$$v_j(r) = \int_{\partial B_r} H_j, \quad v_1(r) = \int_{\partial B_r} H_1. \quad (1)$$

where integration is with respect to the  $(m-1)$ -dimensional Hausdorff measure of  $\partial B_r$ . Fix an equator  $E \subset \mathbb{S}^m$  and suppose that either

$$\begin{aligned} (i) \quad & \int^{+\infty} \frac{dr}{v_j(r)} = +\infty \quad \text{and} \quad H_1 \notin L^1(M) \quad \text{or} \\ (ii) \quad & \int^{+\infty} \frac{dr}{v_j(r)} < +\infty \quad \text{and} \\ & \limsup_{r \rightarrow +\infty} \sqrt{v_1(r)v_j(r)} \int_r^{+\infty} \frac{ds}{v_j(s)} > \frac{1}{2} \left[ (j+1) \binom{m+1}{j+2} H_{j+1} \right]^{-1/2}. \end{aligned} \quad (2)$$

Then, there exists a divergent sequence  $\{x_k\} \subset M$  such that  $\nu(x_k) \in E$ , where  $\nu$  is the spherical Gauss map.

**Remark 1.2.** Up to changing the orientation of  $M$ , we can suppose that the second fundamental form at  $p$  is positive definite. As we will see later in more detail, this has the remarkable consequence that each  $H_i$ ,  $1 \leq i \leq n$ , is strictly positive at every point of  $M$ . In particular,  $v_1$  and  $v_j$  are both strictly positive and the requirements in (2) are meaningful.

**Remark 1.3.** When  $j = 1$ , the existence of an elliptic point  $p \in M$  can be replaced by requiring  $H_2$  to be a positive constant, see [6] for details. The case  $j = 0$  has been considered in [4].

We clarify the role of (i) and (ii) with some examples. First, we deal with the case  $j \neq 1$ , and we assume that  $v_j$  is of order  $r^k$  (resp  $e^{kr}$ ), for some  $k > 0$ . Then assumption (ii) requires that  $v_1(r)$  is of order at least  $r^{k-2}$  (resp  $e^{kr}$ ). Roughly speaking,  $v_1$  has to be big enough with respect to the other integral curvature  $v_j$ . Under additional requirements on the intrinsic curvatures of  $M$ , standard volume comparisons allow to control the volume of  $\partial B_r$  and (ii) can be read as  $H_1$  not decaying too fast at infinity. When  $j = 1$ , things are somewhat different. Indeed, (ii) implies that  $v_1(r)$  does not grow too fast, that is, loosely speaking, it has at most exponential growth. This shows that two opposite effects balances in condition (ii). The same happens for (i) with  $j = 1$ , as a consequence of Cauchy-Schwartz inequality and coarea formula

$$\left( \int_R^r \frac{ds}{v_1(s)} \right) \left( \int_{B_r \setminus B_R} H_1 \right) \geq (r - R)^2.$$

Finally, we stress that (i) and (ii) are mild hypotheses as they only involve the integral of extrinsic curvatures. In other words, no pointwise control is required.

Up to identifying the image of the tangent space at  $p \in M$  with an affine hyperplane of  $\mathbb{R}^{m+1}$  in the standard way, we can also prove the following result:

**Theorem 1.4.** *Let  $f : M \rightarrow \mathbb{R}^{m+1}$  be a hypersurface with  $H_{j+1} \equiv 0$ . If  $j \geq 1$ , assume  $\text{rank}(A) > j$  at every point. Define  $v_1, v_j$  as in (1). Then, under assumptions (2) (i) or (ii), for every compact set  $K \subset M$  we have*

$$\bigcup_{p \in M \setminus K} T_p M \equiv \mathbb{R}^{m+1},$$

*that is, the tangent envelope of  $M \setminus K$  coincides with  $\mathbb{R}^{m+1}$ .*

**Remark 1.5.** As we will see later, condition  $\text{rank}(A) > j$  implies that  $H_i > 0$  for every  $1 \leq i \leq j$ .

## 2 Preliminaries

We start recalling the definition and some properties of the Newton tensors  $P_j$ ,  $j \in \{0, \dots, m\}$ . They are inductively defined by

$$P_0 = I, \quad P_j = S_j I - A P_{j-1}.$$

For future use, we state the following algebraic lemma. For a proof, see [3].

**Lemma 2.1.** *Let  $\{e_i\}$  be the principal directions associated with  $A$ ,  $Ae_i = k_i e_i$ , and let  $S_j(A_i)$  be the  $j$ -th symmetric function of  $A$  restricted to the  $(m-1)$ -dimensional space  $e_i^\perp$ . Then, for each  $1 \leq j \leq m-1$ ,*

- (1)  $AP_j = P_j A$ ;
- (2)  $P_j e_i = S_j(A_i) e_i$ ;
- (3)  $\text{Tr}(P_j) = \sum_i S_j(A_i) = (m-j)S_j$ ;
- (4)  $\text{Tr}(AP_j) = \sum_i k_i S_j(A_i) = (j+1)S_{j+1}$ ;
- (5)  $\text{Tr}(A^2 P_j) = \sum_i k_i^2 S_j(A_i) = S_1 S_{j+1} - (j+2)S_{j+2}$ .

It follows from (2) in the above lemma, and from the definition of  $P_m$  that  $P_m = 0$ . Related to the  $j$ -th Newton tensor there is a well defined, symmetric differential operator acting on  $C_c^\infty(M)$ :

$$L_j u = \text{Tr}(P_j \text{Hess}(u)) = \text{div}(P_j \nabla u) \quad \forall u \in C_c^\infty(M), \quad (3)$$

where the last equality is due to the fact that  $A$  is a Codazzi tensor in  $\mathbb{R}^{m+1}$ , see [5], [13].  $L_j$  naturally appears when looking for stationary points of the curvature integral

$$\mathcal{A}_j(M) = \int_M S_j dV_M,$$

for compactly supported volume preserving variations. These functionals can be viewed as a generalization of the volume functional. In fact, in [3] and [6] the stationary points of  $\mathcal{A}_j$  are characterized as those immersions having constant  $S_{j+1}$ . In the above mentioned paper [6], M.F. Elbert computes the second variation of  $\mathcal{A}_j$  in more general ambient spaces and obtains in the Euclidean setting the expression

$$T_j = L_j + (S_1 S_{j+1} - (j+2)S_{j+2})$$

for the Jacobi operator. In what follows we are interested in the case of  $L_j$  elliptic. There are a number of different results giving sufficient conditions to guarantee this fact, and the next two fit the situation of our main theorems.

**Proposition 2.2.** *Let  $M$  be an  $m$ -dimensional connected, orientable hypersurface of some space form  $N$ . Then,  $L_i$  is elliptic for every  $1 \leq i \leq j$  in each of the following cases:*

- (i)  *$M$  contains an elliptic point, that is, a point  $p \in M$  at which  $A$  is definite (positive or negative), and  $S_{j+1} \neq 0$  at every point of  $M$ . Note that, up to changing the orientation of  $M$ , we can assume  $A_p$  to be positive definite, and by continuity  $S_{j+1} > 0$  on  $M$ .*
- (ii)  *$S_{j+1} \equiv 0$  and  $\text{rank}(A) > j$  at every point of  $M$ .*

Moreover, in both cases, every  $i$ -mean curvature  $H_i$  is strictly positive on  $M$ , for  $1 \leq i \leq j$ .

For a proof of (i) see [3], while for (ii) see [10].

From the above proposition, the requirements on  $p$  and  $\text{rank}(A)$  in the main theorems ensure ellipticity. As stressed in Remark 1.3, when  $j = 2$  in [6] it is shown that the sole requirement  $H_2 > 0$  implies the ellipticity of  $L_1$ . In the assumptions of the above proposition, we can define the  $j$ -volume of some measurable subset  $K \subset M$  as the integral

$$\mathcal{A}_j(K) = \int_K S_j dV_M.$$

Hereafter, we restrict to the case  $L_j$  elliptic. Given the relatively compact domain  $\Omega \subset M$ ,  $L_j$  is bounded from below on  $C_c^\infty(\Omega)$  and, by Rellich theorem, for a sufficiently large  $\lambda$ ,  $(L_j - \lambda)$  is invertible with compact resolvent. By standard spectral theory,  $L_j$  is therefore essentially self-adjoint on  $C_c^\infty(\Omega)$  (Theorem 3.3.2 in [12]). Essential self-adjointness implies that  $C_c^\infty(\Omega)$  and  $\text{Lip}_0(\Omega)$  are cores for the quadratic form associated to  $L_j$ . The first eigenvalue  $\lambda_1^{T_j}(\Omega)$ , with Dirichlet boundary condition, is therefore defined by the Rayleigh characterization

$$\lambda_1^{T_j}(\Omega) = \inf_{\substack{\phi \in \text{Lip}_0(\Omega) \\ \phi \neq 0}} \frac{\int_\Omega \langle P_j(\nabla \phi), \nabla \phi \rangle - \int_\Omega (S_1 S_{j+1} - (j+2) S_{j+2}) \phi^2}{\int_\Omega \phi^2},$$

where  $\text{Lip}_0(\Omega)$  can be replaced with  $C_c^\infty(\Omega)$ . By the monotonicity property of eigenvalues (or, in other words, since  $L_j$  satisfies the unique continuation property, [2]), if  $\Omega_1$  is a domain with compact closure in  $\Omega_2$ , and  $\Omega_2 \setminus \Omega_1$  has nonempty interior,  $\lambda_1^{T_j}(\Omega_1) > \lambda_1^{T_j}(\Omega_2)$ . Hence, we deduce the existence of

$$\lambda_1^{T_j}(M) = \lim_{\mu \rightarrow +\infty} \lambda_1^{T_j}(\Omega_\mu),$$

where  $\{\Omega_\mu\}$  is any exhaustion of  $M$  by means of increasing, relatively compact domains with smooth boundary. The next result is substantially an application of the result of Moss-Piepenbrink [11], slightly modified according to Fischer-Colbrie and Schoen [8] and Fischer-Colbrie [7] (consult also [12], Chapter 3 and, for the case of  $L_1$ , [6]).

**Proposition 2.3.** *Let  $M$  be a Riemannian manifold and let  $T_j$  be as above. The following statements are equivalent:*

- (i)  $\lambda_1^{T_j}(M) \geq 0$ ;
- (ii) *there exists  $u \in C^\infty(M)$ ,  $u > 0$  solution of  $T_j u = 0$  on  $M$ .*

Furthermore, there exists a compact set  $K \subset M$  and  $u \in C^\infty(M \setminus K)$ ,  $u > 0$  solution of  $T_j u = 0$  on  $M \setminus K$  if and only if  $\lambda_1^{T_j}(M \setminus K) \geq 0$ .

Next, we shall need to consider the following Cauchy problem; here, as usual,  $\mathbb{R}^+ = (0, +\infty)$  and  $\mathbb{R}_0^+ = [0, +\infty)$ .

$$\begin{cases} (v(t)z'(t))' + A(t)v(t)z(t) = 0 & \text{on } \mathbb{R}^+ \\ z'(t) = O(1) \text{ as } t \downarrow 0^+, \quad z(0^+) = z_0 > 0 \end{cases} \quad (4)$$

where  $A(t)$  and  $v(t)$  satisfy the following conditions:

(A1)  $A(t) \in L_{\text{loc}}^\infty(\mathbb{R}_0^+)$ ,  $A(t) \geq 0$ ,  $A \not\equiv 0$  in  $L_{\text{loc}}^\infty$  sense;

(V1)  $v(t) \in L_{\text{loc}}^\infty(\mathbb{R}_0^+)$ ,  $v(t) \geq 0$ ,  $\frac{1}{v(t)} \in L_{\text{loc}}^\infty(\mathbb{R}^+)$ ;

(V2) there exists  $a \in \mathbb{R}^+$  such that  $v$  is increasing on  $(0, a)$  and  $\lim_{t \rightarrow 0^+} v(t) = 0$ .

Observe that (V2) has to be interpreted as there exists a version of  $v$  which is increasing near 0 and whose limit as  $t \rightarrow 0^+$  is 0.

By Proposition A.1 of [4] under the above assumptions (4) has a solution  $z(t) \in \text{Lip}_{\text{loc}}(\mathbb{R}_0^+)$  (and condition  $z'(t) = O(1)$  as  $t \downarrow 0^+$  is satisfied in an appropriate sense). Furthermore by Proposition A.3 of [4],  $z(t)$  has only isolated zeros. In case  $1/v \in L^1((1, +\infty))$ , by Proposition 2.5 of [4] if, for some  $T > 0$ ,

$$\limsup_{t \rightarrow \infty} \frac{\int_T^t \sqrt{A(s)} ds}{-\frac{1}{2} \log \int_t^{+\infty} \frac{ds}{v(s)}} > 1 \quad (5)$$

then, every solution of

$$\begin{cases} (v(t)z'(t))' + A(t)v(t)z(t) = 0 & \text{on } (t_0, +\infty), \quad t_0 > 0 \\ z(t_0) = z_0 > 0 \end{cases} \quad (6)$$

has isolated zeros and is oscillatory. The same happens if

$$\int^{+\infty} \frac{dt}{v(t)} = +\infty \quad \text{and} \quad \int^{+\infty} A(t)v(t)dt = +\infty. \quad (7)$$

(see Corollary 2.4 of [4]).

A final result that we shall use is the following computation. (For a proof see [13], [1]).

**Proposition 2.4.** *Let  $f : M \rightarrow \mathbb{R}^{m+1}$  be an isometric immersion of an oriented hypersurface and  $\nu : M \rightarrow \mathbb{S}^m$  its Gauss map. Fix  $a \in \mathbb{S}^m$ . Then*

$$L_j \langle a, \nu \rangle = -(S_1 S_{j+1} - (j+2) S_{j+2}) \langle a, \nu \rangle - \langle \nabla S_{j+1}, a \rangle;$$

$$L_j \langle f, \nu \rangle = -(j+1) S_{j+1} - (S_1 S_{j+1} - (j+2) S_{j+2}) \langle f, \nu \rangle - \langle \nabla S_{j+1}, f \rangle. \quad (8)$$

where  $\langle, \rangle$  stands for the scalar product of vectors in  $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ .

In particular, if  $S_{j+1}$  is constant, we have  $T_j\langle a, \nu \rangle = 0$ . Moreover, if  $S_{j+1} \equiv 0$ ,  $T_j\langle f, \nu \rangle \equiv 0$ .

### 3 Proof of Theorem 1.1

Fix an equator  $E$  and reason by contradiction: assume that there exists a sufficiently large geodesic ball  $B_R$  such that, outside  $B_R$ ,  $\nu$  does not meet  $E$ . In other words,  $\nu(M \setminus B_R)$  is contained in the open spherical caps determined by  $E$ . Indicating with  $a \in \mathbb{S}^m$  one of the two focal points of  $E$ ,  $\langle a, \nu(x) \rangle \neq 0$  on  $M \setminus B_R$ .

Let  $\mathcal{C}$  be one of the (finitely many) connected components of  $M \setminus B_R$ ; then  $\nu(\mathcal{C})$  is contained in only one of the open spherical caps determined by  $E$ . Up to replacing  $a$  with  $-a$ , we can suppose  $u = \langle a, \nu \rangle > 0$  on  $\mathcal{C}$ . Proceeding in the same way for each connected component we can construct a positive function  $u$  on  $M \setminus B_R$ . Since  $S_{j+1}$  is constant, by Proposition 2.4 we have that  $u > 0$  satisfies

$$T_j u = L_j u + (S_1 S_{j+1} - (j+2)S_{j+2})u = 0$$

on  $M \setminus B_R$ . Thus, by Proposition 2.3,  $\lambda_1^{T_j}(M \setminus B_R) \geq 0$ . We shall now show that the assumptions of the theorem contradict this fact. As already stressed, the existence of an elliptic point forces both  $H_j$  and  $H_{j+1}$  to be positive. Fix a radius  $0 < R_0 < R$  and let  $K_j$  be a smooth positive function on  $M$  such that

$$K_j(x) = \begin{cases} 1 & \text{on } B_{R_0/2} \\ (m-j)S_j & \text{on } M \setminus B_{R_0} \end{cases} \quad (9)$$

Next, we define

$$v_j(t) = \int_{\partial B_t} K_j \quad (10)$$

Using Proposition 1.2 of [4] we see that  $v_j(t)$  satisfies (V1) with  $v_j(t) > 0$  on  $\mathbb{R}^+$  and (V2). Next, we define

$$A(t) = \frac{1}{v_j(t)} \int_{\partial B_t} S_1 S_{j+1} - (j+2)S_{j+2}. \quad (11)$$

Then, repeated applications of Newton inequalities give

$$H_1 H_{j+1} - H_{j+2} \geq 0. \quad (12)$$



Thus, using (12)

$$\begin{aligned}
S_1 S_{j+1} - (j+2) S_{j+2} &= m \binom{m}{j+1} H_1 H_{j+1} - (j+2) \binom{m}{j+2} H_{j+2} = \\
&= \binom{m}{j+1} (m H_1 H_{j+1} - (m-j-1) H_{j+2}) \\
&\geq \binom{m}{j+1} \left[ m - \frac{m-j-1}{j+2} \right] H_1 H_{j+1} = (j+1) \binom{m+1}{j+2} H_1 H_{j+1} \geq 0.
\end{aligned} \tag{13}$$

This implies  $A(t) \geq 0$ , and

$$A(t) v_j(t) \geq (j+1) \binom{m+1}{j+2} H_{j+1} \int_{\partial B_t} H_1 = (j+1) \binom{m+1}{j+2} H_{j+1} v_1(t).$$

If  $1/v_j \notin L^1((1, +\infty))$ , then under (2) (i) and by the coarea formula we deduce  $Av_j \notin L^1(\mathbb{R}^+)$ . Hence, we can apply (7) to deduce that every solution of

$$\begin{cases} (v_j(t) z'(t))' + A(t) v_j(t) z(t) = 0 & \text{on } (t_0, +\infty), t_0 > 0 \\ z(t_0) = z_0 > 0 \end{cases} \tag{14}$$

is oscillatory. The same conclusion holds when  $1/v_j \in L^1((1, +\infty))$ . Indeed, from (11), (13)

$$\frac{\int_T^t \sqrt{A(s)} ds}{-\frac{1}{2} \log \int_t^{+\infty} \frac{ds}{v_j(s)}} \geq 2 \sqrt{(j+1) \binom{m+1}{j+2} H_{j+1}} \frac{\int_T^t \sqrt{\frac{v_1(s)}{v_j(s)}} ds}{-\log \int_t^{+\infty} \frac{ds}{v_j(s)}}. \tag{15}$$

Using De l'Hopital rule and (2) (ii), (5) is met. Let now  $R < T_1 < T_2$  be two consecutive zeros of  $z(t)$  after  $R$ . Define

$$\psi(x) = \begin{cases} z(r(x)) & \text{on } \overline{B_{T_2}} \setminus B_{T_1} \\ 0 & \text{outside } \overline{B_{T_2}} \setminus B_{T_1}. \end{cases}$$

Note that  $\psi \equiv 0$  on  $\partial(\overline{B_{T_2}} \setminus B_{T_1})$ ,  $\psi \in \text{Lip}_0(M)$  and  $\nabla \psi(x) = z'(r(x)) \nabla r(x)$  where defined. Furthermore, by the coarea formula and the definition of  $A(t)$  we have

$$\begin{aligned}
\int_M (S_1 S_{j+1} - (j+2) S_{j+2}) \psi^2 &= \int_{T_1}^{T_2} z^2(t) \int_{\partial B_t} (S_1 S_{j+1} - (j+2) S_{j+2}) dt = \\
&= \int_{T_1}^{T_2} z^2(t) A(t) v_j(t) dt = (m-j) \int_M S_j A(r) \psi^2.
\end{aligned}$$

Thus, using (4), the above identity and again the coarea formula

$$\begin{aligned}
& \int_M \langle P_j(\nabla\psi), \nabla\psi \rangle - (S_1 S_{j+1} - (j+2)S_{j+2})\psi^2 \\
& \leq \int_M \text{Tr}(P_j)|\nabla\psi|^2 - (S_1 S_{j+1} - (j+2)S_{j+2})\psi^2 = \\
& = \int_M (m-j)S_j|\nabla\psi|^2 - (S_1 S_{j+1} - (j+2)S_{j+2})\psi^2 \\
& = (m-j) \int_{\overline{B_{T_2}} \setminus B_{T_1}} S_j[(z')^2 - A(t)z^2] = \\
& = (m-j) \int_{T_1}^{T_2} [(z')^2 - A(t)z^2]v_j(t)dt = \\
& = (m-j) \left\{ z(t)z'(t)v_j(t) \Big|_{T_1}^{T_2} - \int_{T_1}^{T_2} [(v_j(t)z'(t))' + A(t)v_j(t)z(t)]z(t)dt \right\} = 0.
\end{aligned}$$

It follows that

$$\lambda_1^{T_j}(\overline{B_{T_2}} \setminus B_{T_1}) \leq \frac{1}{\int_M \psi^2} \left\{ \int_M \langle P_j(\nabla\psi), \nabla\psi \rangle - (S_1 S_{j+1} - (j+2)S_{j+2})\psi^2 \right\} = 0.$$

As a consequence  $\lambda_1^{T_j}(M \setminus B_R) < 0$ , which gives the desired contradiction.

**Remark 3.1.** As a matter of fact, the orientability of  $M$  is not needed. If  $M$  is non orientable,  $\nu$  is not globally defined. However, changing the sign of  $\nu$  does not change either the assumptions or the conclusion of Theorem 1.1, since the antipodal map on  $\mathbb{S}^m$  leaves each  $E$  fixed. If  $\langle a, \nu \rangle \neq 0$  on  $M \setminus B_R$ , the normal field  $X = \langle a, \nu \rangle \nu$  is nowhere vanishing and globally defined on  $M \setminus B_R$ . This shows that, in any case, every connected component of  $M \setminus B_R$  is orientable.

## 4 Proof of Theorem 1.4

Assume that, for some  $K$ , the tangent envelope of  $M \setminus K$  does not coincide with  $\mathbb{R}^{m+1}$ . By choosing cartesian coordinates appropriately, we can assume

$$0 \notin \bigcup_{p \in M \setminus K} T_p M.$$

Then, the function  $u = \langle f, \nu \rangle$  is nowhere vanishing and smooth on  $M \setminus K$ . Up to changing the orientation,  $u > 0$  on  $M \setminus K$ . By Proposition 2.4,  $T_j u = -(j+1)S_{j+1}u = 0$ . Note that here the assumption  $H_{j+1} \equiv 0$  is essential. It follows that  $\lambda_1^{T_j}(M \setminus K) \geq 0$ . The rest of the proof is identical to that of Theorem 1.1. Again, according to Remark 3.1 we can drop the orientability assumption on  $M$ . Indeed, if the tangent envelope of  $M \setminus K$  does not cover  $\mathbb{R}^{m+1}$ , the vector field  $X = \langle f, \nu \rangle \nu$  is a globally defined, nowhere vanishing normal vector field on  $M \setminus K$ , hence  $M \setminus K$  is orientable.

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